Differential invariants for symplectic Lie algebras realized by boson operators

2004 J. Phys. A: Math. Gen. 37 4797

(http://iopscience.iop.org/0305-4470/37/17/010)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 4797–4812 PII: S0305-4470(04)61348-5

Differential invariants for symplectic Lie algebras realized by boson operators

Marconi Soares Barbosa and Esmerindo de Sousa Bernardes

Departamento de Física e Ciência dos Materiais, Instituto de Física de São Carlos, Universidade de São Paulo, Caixa Postal 369, 13560.970 São Carlos, SP, Brazil

E-mail: sousa@if.sc.usp.br

Received 11 March 2003
Published 14 April 2004
Online at stacks.iop.org/JPhysA/37/4797 (DOI: 10.1088/0305-4470/37/17/010)

Abstract

The study of algebraic symplectic models in high energy physics and even in molecular genetics has received much attention in the past decades. In this paper, we first survey one known bosonic realization with N types of bosons operators in a $(2j+1)$-dimensional space (with j being the semi-integer). Then, using the general theory of differential invariants, we determine two sets of sp$(2j+1)$-differential invariants. In one set, when $N \geqslant 2j+1$, all independent invariants of any order are obtained in a simple closed form. When $N < 2j+1$, we could present only particular matrix forms. In the latter case, the differential invariants up to the second order are explicitly presented for the symplectic algebras sp(2), sp(4) and sp(6) realized by up to three types of bosons. As applications, we present the simplest sp(2)-invariant Lagrangians, null-Lagrangians, new solutions to the Helmholtz inverse problem and evolution equations, which will be of fundamental importance in the construction of dynamical systems invariant under the symplectic group.

PACS numbers: 02.20.Sv, 03.65.Fd

1. Introduction

The essence of equivalence is the determination of when two mathematical objects, such as differential equations, can be identified under a change of variables. Conditions guaranteeing equivalence are frequently required and most effectively formulated in terms of invariants, which are quantities unaffected by the changes of variables or by the action of transformation groups [1, 2]. Such general topics arise naturally in mathematical methods of physics and are essential tools in many models of modern physics [3]. In a physical model where the observable properties are governed by a system of differential equations, symmetry requirements are postulated in many cases. As differential invariants completely characterize invariant systems of differential equations [1, 4], they are the building blocks of physical theories where some form of invariance shall appear in the basic differential equations. Besides this important issue, where the usual application of differential invariants is useful in providing sets of new invariant nonlinear differential equations and to characterize invariant variational problems, new applications in the area of image processing and computer vision are beginning to occur in the literature [5, 6].

In physics, many important symmetry properties are described by Lie groups [7–10] or, more precisely, by their associated Lie algebras, which can be used to write the differential equations of a specific model. To accomplish that, it is necessary to realize the required Lie algebra by differential operators specially designed for each model. Once the realization is given, its differential invariants can be computed by either tools of the prolongation theory [1, 11], or more specific methods [12, 13].

While the calculation of differential invariants has been fostered by applications related to both orthogonal groups [13] and the Poincaré group [12], the same implementation for symplectic algebras has been overlooked despite its importance to classical mechanics, optics and quantum physics [14–16]. The main goal in this paper is to add more studies about symplectic differential invariants (see [11] for a recent contribution), which will be the foundations of dynamical systems invariant under the symplectic group.

The symplectic algebra sp(6) has been recently applied in molecular biology, where an algebraic model is under consideration to study the genetic code (evolution, functionality, etc) [17–19]. Regarding this model, Chacón and Moshinsky have realized the sp(6) algebra with two types of bosons as a subalgebra of u(6) and stressed that the boson operators 'may have some significance as fundamental blocks in the genetic code' [20]. This same algebra, although realized by boson operators in a different way, is also used in algebraic models developed in both nuclear and particle physics [21–26]. Therefore, establishing a dynamical system with a prescribed symplectic symmetry is fundamental to algebraic models using symplectic algebras. Naturally, the first step in that direction is the characterization of their differential invariants and the identification of Lagrangians (locally) invariant under the symplectic group.

The differential invariants obtained in this work yield an explicit functional basis for locally invariant partial differential equations, whose solutions are related by symplectic operations. This new class of nonlinear partial differential equations with a predefined symmetry property not only offers its well-known symmetry-related operational benefits but also restricts possible dynamical systems and variational problems with such symmetry. Indeed, the construction of a dynamical system with a prescribed symmetry and adapted to a specific Lie group chain is a challenging open problem in physics.

This paper is organized as follows: in section 2, the symplectic algebras $\mathrm{sp}(2j+1)$ are realized by N types of boson operators in a $(2j+1)$-dimensional space. Section 3 presents the symplectic vector fields prolonged in a jet space spanned by one dependent variable (or scalar field) and its high-order derivatives. The symplectic differential invariants are obtained, both in a closed form when $N \geqslant 2j+1$, due to the existence of an invariant contact coframe, and in matricial forms when $N < 2j+1$. In the latter case, the differential invariants up to second order are explicitly presented for the symplectic algebras sp(2), sp(4) and sp(6) realized by up to three types of bosons. In section 4, we present examples of sp(2)-invariant Lagrangians and evolution equations as well as null-Lagrangians and new solutions to the Helmholtz inverse problem of the calculus of variations. Our concluding remarks are given in section 5.

2. A bosonic realization for symplectic algebras

Lie algebras are frequently realized by boson operators and a polynomial basis in the creation operators is desired to build their irreducible representations. When there is only one type of boson, only the symmetric and anti-symmetric irreducible representations admit a polynomial basis. However, most applications in physics also need the mixed irreducible representations which require more than one type of boson.

Consider N particles in a $(2j+1)$-dimensional space, with j being the semi-integer. Let $x_{\alpha m}$ and $p_{\alpha m}$ be the components of the coordinates and momenta of each particle, respectively. Let Greek indices indicate particles ($\alpha = 1, 2, \dots, N$) and Latin indices indicate vector components in spherical form ($m = -j, \dots, j$) with the symplectic metric (see [27, chapter 24] and [28, chapters 5 and 6])

$$g_{mm'} = (-1)^{j+m}\delta_{m\bar{m}'} \qquad g^{mm'} = -g_{mm'} = g_{m'm} \qquad \bar{m} = -m. \tag{1}$$

Therefore, the corresponding coordinates and momenta in their contravariant form are given as

$$x_{\alpha}^{m} = (-1)^{j+m}x_{\alpha\bar{m}} \qquad p_{\alpha}^{m} = (-1)^{j+m}p_{\alpha\bar{m}}. \tag{2}$$

Let us introduce, respectively, creation and annihilation operators (also known as bosonic operators) by the usual definitions [29, 30]

$$a_{\alpha m}^{\dagger} = \tfrac{1}{\sqrt{2}}(x_{\alpha m} - \mathrm{i}p_{\alpha m}) \qquad a_{\alpha}^{m} = \tfrac{1}{\sqrt{2}}\left(x_{\alpha}^{m} + \mathrm{i}p_{\alpha}^{m}\right). \tag{3}$$

Then, the usual commutation relations for the bosonic operators forming the Weyl algebra

$$\left[a_{\alpha}^{m}, a_{\alpha'm'}^{\dagger}\right] = \delta_{m'}^{m}\delta_{\alpha\alpha'} \qquad \left[a_{\alpha}^{m}, a_{\alpha'}^{m'}\right] = \left[a_{\alpha m}^{\dagger}, a_{\alpha'm'}^{\dagger}\right] = 0 \tag{4}$$

can be obtained from the well-known quantum relations ($\hbar = 1$) [29, 30]

$$\left[x_{\alpha}^{m}, p_{\alpha'm'}\right] = \mathrm{i}\delta_{m'}^{m}\delta_{\alpha\alpha'} \tag{5}$$

where all other commutators are zero. It is a fact that unitary algebras can be realized by bosonic operators [31]. Indeed, not only the algebra but also their irreducible representations can be realized by polynomial expressions in the creation $a_{\alpha m}^{\dagger}$ operators [20, 32–34], which, together with the commutation relations (4), allow us to interpret the annihilation operators a_{α}^{m} as the differential operators

$$a_{\alpha}^{m} = \frac{\partial}{\partial a_{\alpha m}^{\dagger}} \tag{6}$$

when they are acting on polynomials in the creation operators. Also, in order to simplify the notation, let

$$\eta_{\alpha m} = a_{\alpha m}^{\dagger} \qquad \xi_{\alpha}^{m} = a_{\alpha}^{m} = \frac{\partial}{\partial \eta_{\alpha m}}. \tag{7}$$

Now, the symplectic algebras $\mathrm{sp}(2j+1)$, with j being semi-integers only, can be realized as [20]

$$\mathcal{L}_{m'}^{m} = \mathcal{C}_{m'}^{m} + (-1)^{m+m'}\mathcal{C}_{\bar{m}}^{\bar{m}'} \qquad \mathcal{L}_{\bar{m}'}^{\bar{m}} = (-1)^{m+m'}\mathcal{L}_{m}^{m'} \tag{8}$$

where

$$\mathcal{C}_{m}^{m'} = \sum_{\alpha=1}^{N}\eta_{\alpha m}\xi_{\alpha}^{m'} \tag{9}$$

realizes the unitary algebra $\mathrm{u}(2j+1)$. Note that the vector fields (8) are horizontal, i.e. they do not depend on the derivatives of the dependent variable. Although our interest lies only in the symplectic elements, it is important to mention that the contraction

$$C_{\alpha\beta} = \sum_{m=-j}^{j}\eta_{\alpha m}\xi_{\beta}^{m} \tag{10}$$

Table 1. Polynomial basis with two types of bosons to the adjoint irreducible representation [2, 0] of sp(4) adapted to the chain (A.1). The roots (first three columns) are given in three different bases [43, 39]: FWS (fundamental weight system), DYN (Dynkin, formed by the basic irreducible representations) and SRS (simple root system), respectively. The fourth column is the 'quantum numbers' of each vector of the adjoint representation. The last column presents the correspondence between the notation of [44] and the current notation shown in the fifth column.

FWS	DYN	SRS	$\vert\sigma_1, \sigma_2, h_1, h_2\rangle$	Polynomial basis	$\mathcal{L}_m^{m'}$	
[2, 0]	(2, 0)	{2, 1}	$\vert 2, 0, 2, 0\rangle$	$\eta_{13}\eta_{15}\eta_{23} - \eta_{13}\eta_{13}\eta_{25}$	$\mathcal{L}_3^{\bar{3}}$	$-\sqrt{2}E_4^+$
[−2, 0]	(−2, 0)	{−2, −1}	$\vert 2, 0, -2, 0\rangle$	$\eta_{1\bar{3}}\eta_{15}\eta_{2\bar{3}} - \eta_{1\bar{3}}\eta_{1\bar{3}}\eta_{25}$	$\mathcal{L}_{\bar{3}}^{3}$	$-\sqrt{2}E_4^-$
[1, 1]	(0, 1)	{1, 1}	$\vert 1, 1, 1, 1\rangle$	$+\eta_{11}\eta_{15}\eta_{23} + \eta_{13}\eta_{15}\eta_{21} - 2\eta_{11}\eta_{13}\eta_{25}$	$\mathcal{L}_1^{\bar{3}}$	$-E_3^+$
[−1, −1]	(0, −1)	{−1, −1}	$\vert 1, 1, -1, -1\rangle$	$-\eta_{1\bar{1}}\eta_{15}\eta_{2\bar{3}} - \eta_{1\bar{3}}\eta_{15}\eta_{2\bar{1}} + 2\eta_{1\bar{1}}\eta_{1\bar{3}}\eta_{25}$	$\mathcal{L}_{\bar{3}}^{1}$	$-E_3^-$
[1, −1]	(2, −1)	{1, 0}	$\vert 1, 1, 1, -1\rangle$	$+\eta_{1\bar{1}}\eta_{15}\eta_{23} + \eta_{13}\eta_{15}\eta_{2\bar{1}} - 2\eta_{1\bar{1}}\eta_{13}\eta_{25}$	$\mathcal{L}_3^{1}$	E_1^+
[−1, 1]	(−2, 1)	{−1, 0}	$\vert 1, 1, -1, 1\rangle$	$-\eta_{11}\eta_{15}\eta_{2\bar{3}} - \eta_{1\bar{3}}\eta_{15}\eta_{21} + 2\eta_{11}\eta_{1\bar{3}}\eta_{25}$	$\mathcal{L}_1^{3}$	E_1^-
[0, 2]	(−2, 2)	{0, 1}	$\vert 0, 2, 0, 2\rangle$	$\eta_{11}\eta_{15}\eta_{21} - \eta_{11}\eta_{11}\eta_{25}$	$\mathcal{L}_1^{\bar{1}}$	$\sqrt{2}E_2^+$
[0, −2]	(2, −2)	{0, −1}	$\vert 0, 2, 0, -2\rangle$	$\eta_{1\bar{1}}\eta_{15}\eta_{2\bar{1}} - \eta_{1\bar{1}}\eta_{1\bar{1}}\eta_{25}$	$\mathcal{L}_{\bar{1}}^{1}$	$\sqrt{2}E_2^-$
[0, 0]	(0, 0)	{0, 0}	$\vert 0, 0, 0, 0\rangle$	$-\eta_{13}\eta_{15}\eta_{2\bar{3}} - \eta_{1\bar{3}}\eta_{15}\eta_{23} + 2\eta_{13}\eta_{1\bar{3}}\eta_{25}$	$\mathcal{L}_3^{3}$	H_1
[0, 0]	(0, 0)	{0, 0}	$\vert 0, 2, 0, 0\rangle$	$+\eta_{11}\eta_{15}\eta_{2\bar{1}} + \eta_{1\bar{1}}\eta_{15}\eta_{21} - 2\eta_{11}\eta_{1\bar{1}}\eta_{25}$	$\mathcal{L}_{\bar{1}}^{1}$	H_2

realizes the unitary algebra u(N). These realizations are equivalent to those given by Schwinger's technique using the fundamental irreducible representations [31]. From (4), the commutation relations for the unitary algebras are

$$\left[\mathcal{C}_m^{m'}, \mathcal{C}_{m''}^{m'''}\right] = \delta_{m''}^{m'}\mathcal{C}_m^{m'''} - \delta_m^{m'''}\mathcal{C}_{m''}^{m'} \qquad [C_{\alpha\beta}, C_{\alpha'\beta'}] = \delta_{\alpha'\beta}C_{\alpha\beta'} - \delta_{\alpha\beta'}C_{\alpha'\beta}. \tag{11}$$

Consequently, the commutation relations for the symplectic algebras are

$$\left[\mathcal{L}_m^{m'}, \mathcal{L}_{m''}^{m'''}\right] = \delta_{m''}^{m'}\mathcal{L}_m^{m'''} - \delta_m^{m'''}\mathcal{L}_{m''}^{m'} + (-1)^{m''+m'''}\left(\delta_{\bar{m}'''}^{m'}\mathcal{L}_m^{\bar{m}''} - \delta_m^{\bar{m}''}\mathcal{L}_{\bar{m}'''}^{m'}\right). \tag{12}$$

The Casimir operator of sp($2j+1$) can be written as

$$\mathfrak{C}_{\mathrm{sp}(2j+1)} = \sum_{m=1/2}^{j}\left(\mathcal{L}_m^m\right)^2 + \frac{1}{2}\sum_{m=1/2}^{j}\left[\mathcal{L}_m^{\bar{m}}, \mathcal{L}_{\bar{m}}^{m}\right]_+ + \sum_{0<m<|m'|}^{j}\left[\mathcal{L}_m^{m'}, \mathcal{L}_{m'}^{m}\right]_+ \tag{13}$$

where $[a, b]_+ = ab + ba$. This second-order invariant operator commutes with every element of any symplectic algebra

$$\left[\mathfrak{C}_{\mathrm{sp}(2j+1)}, \mathcal{L}_m^{m'}\right] = 0. \tag{14}$$

Appendix A gives one possible identification of the bosonic operators $\mathcal{L}_m^{m'}$ with the roots of sp(2) and sp(4). It also shows the polynomial basis in the creation operators ($N = 2$) of two irreducible representations (see tables 1 and 2). It should be mentioned that a polynomial basis in the creation operators for an arbitrary irreducible representation labelled by the highest weight ($\lambda_1, \ldots, \lambda_r$), where r is the rank of sp($2r$), can be constructed realizing the symplectic algebras sp($2j+1$) at least with a number $N = r = j + 1/2$ of different types of bosons.

3. The differential invariants for symplectic algebras

The basic tools of the prolongation theory, which belong to the general theory of differential invariants, can be found in many textbooks [1, 4, 35–37]. In all the following cases, one (smooth) dependent variable $\phi = \phi(\eta)$ in the creation operators η is considered because the following results can be easily generalized to an arbitrary number of dependent variables.

Table 2. Polynomial basis with two types of bosons to the fundamental irreducible representation [1, 0] of sp(4) adapted to the chain (A.1). The corresponding weight system is shown in the first three columns in three different bases [39, 43], as already explained in the caption of table 1.

FWS	DYN	SRS	$\|\sigma_1, \sigma_2, h_1, h_2\rangle$	Polynomial basis
$[1, 0]$	$(1, 0)$	$\{1, \frac{1}{2}\}$	$\|1, 0, 1, 0\rangle$	$+\eta_{15}\eta_{15}\eta_{23} - \eta_{13}\eta_{15}\eta_{25}$
$[0, 1]$	$(-1, 1)$	$\{0, \frac{1}{2}\}$	$\|0, 1, 0, 1\rangle$	$+\eta_{15}\eta_{15}\eta_{21} - \eta_{11}\eta_{15}\eta_{25}$
$[0, -1]$	$(1, -1)$	$\{0, -\frac{1}{2}\}$	$\|1, 0, 0, -1\rangle$	$+\eta_{15}\eta_{15}\eta_{2\bar{1}} - \eta_{1\bar{1}}\eta_{15}\eta_{25}$
$[-1, 0]$	$(-1, 0)$	$\{-1, -\frac{1}{2}\}$	$\|0, 1, -1, 0\rangle$	$-\eta_{15}\eta_{15}\eta_{2\bar{3}} + \eta_{1\bar{3}}\eta_{15}\eta_{25}$

Let X be the Euclidean space of dimension $p = N(2j+1)$, with j semi-integer, whose coordinates consist of p independent creation operators $\eta_{\alpha m}, \alpha \leqslant N$ and $m \leqslant |j|$, and let $U^{(n)}$ be the Euclidean space of dimension $q^{(n)} = \binom{p+n}{n}$, whose coordinates consist of one dependent variable ϕ and its derivatives

$$\phi_{\alpha}^{m} = \xi_{\alpha}^{m}\phi = \frac{\partial\phi}{\partial\eta_{\alpha m}} \qquad \phi_{\alpha\alpha'}^{mm'} = \xi_{\alpha}^{m}\xi_{\alpha'}^{m'}\phi = \frac{\partial^2\phi}{\partial\eta_{\alpha m}\partial\eta_{\alpha' m'}} \qquad \ldots. \tag{15}$$

Let $J^{(n)} = X \times U^{(n)}$ be the nth-order jet space (see [4, chapter 2] and [1, chapter 4] for more details) of dimension

$$\dim J^{(n)} = p + q^{(n)} \qquad q^{(n)} = \binom{p+n}{n} \qquad p = N(2j+1) \tag{16}$$

whose coordinates consist of p independent variables $\eta_{\alpha m}$ and one dependent variable ϕ and its derivatives, (15). Then the nth-order prolongation (see [1, theorem 4.16]) of the vector fields (8) is given in the following theorem.

Theorem 1. *The nth-order prolongation of the vector field $\mathcal{L}_m^{m'}$ defined in (8) is explicitly given by*

$$\mathrm{pr}^{(n)}\mathcal{L}_m^{m'} = \mathcal{L}_m^{m'} + \sum_{m_1=j}^{-j}\sum_{\alpha_1=1}^{N} \varphi_{m\;\alpha_1}^{m'm_1}\widehat{\phi}_{m_1}^{\alpha_1} + \cdots + \sum_{\substack{-j\leqslant m_i\leqslant j\\ 1\leqslant\alpha_i\leqslant N}}{}^{'} \varphi_{m\;\alpha_1\cdots\alpha_n}^{m'm_1\cdots m_n}\widehat{\phi}_{m_1\cdots m_n}^{\alpha_1\cdots\alpha_n} \tag{17}$$

where the operators $\widehat{\phi}_{m\cdots}^{\alpha\cdots}$ are derivatives in the coordinates (15)

$$\widehat{\phi}_{m_1\cdots m_n}^{\alpha_1\cdots\alpha_n} = \frac{\partial}{\partial\phi_{\alpha_1\cdots\alpha_n}^{m_1\cdots m_n}} \tag{18}$$

and the coefficients $\varphi_{m'\;\alpha\cdots}^{m''m\cdots}$ are given by

$$\varphi_{m\;\alpha_1\cdots\alpha_n}^{m'm_1\cdots m_n} = -\left[\delta_m^{m_1}\phi_{\alpha_1\cdots\alpha_n}^{m'\cdots m_2} + \cdots + \delta_m^{m_n}\phi_{\alpha_1\cdots\alpha_n}^{m_1\cdots m'} + (-1)^{(m+m')}\left(\delta_{\bar{m}'}^{m_1}\phi_{\alpha_1\cdots\alpha_n}^{\bar{m}\cdots m_2} + \cdots + \delta_{\bar{m}'}^{m_n}\phi_{\alpha_1\cdots\alpha_n}^{m_1\cdots\bar{m}}\right)\right]. \tag{19}$$

Proof. The characteristics (see [1, definition 4.8]) of the vector fields (8) are

$$Q_m^{m'} = -\mathcal{L}_m^{m'}\phi = \sum_{\alpha=1}^{N}\left[\eta_{\alpha m}\phi_{\alpha}^{m'} + (-1)^{m+m'}\eta_{\alpha\bar{m}'}\phi_{\alpha}^{\bar{m}}\right]. \tag{20}$$

Here, only one dependent variable ϕ is being considered. We can easily verify that the first prolongation coefficient is zero

$$\varphi_m^{m'} = Q_m^{m'} + \mathcal{L}_m^{m'}\phi = 0. \tag{21}$$

Therefore, using a recursive formula (see [1, page 119]), we have

$$\varphi^{m'm_1}_{m\ \alpha_1} = D_{\alpha_1 m_1}\varphi^{m'}_{m} - \sum_{\alpha=1}^{N}\left[(D_{\alpha_1 m_1}\eta_{\alpha m})\phi^{m'}_{\alpha} + (-1)^{m+m'}(D_{\alpha_1 m_1}\eta_{\alpha\bar{m}'})\phi^{\bar{m}}_{\alpha}\right]$$
$$= -\left[\delta^{m_1}_{m}\phi^{m'}_{\alpha_1} + (-1)^{(m+m')}\delta^{m_1}_{\bar{m}'}\phi^{\bar{m}}_{\alpha_1}\right]. \tag{22}$$

Using this last result and the recursive formula once more, we have

$$\varphi^{m'm_1m_2}_{m\ \alpha_1\alpha_2} = D_{\alpha_2 m_2}\varphi^{m'm_1}_{m\ \alpha_1} - \sum_{\alpha=1}^{N}\left[(D_{\alpha_2 m_2}\eta_{\alpha m})\phi^{m'm_1}_{\alpha\ \alpha_1} + (-1)^{m+m'}(D_{\alpha_2 m_2}\eta_{\alpha\bar{m}'})\phi^{\bar{m}m_1}_{\alpha\alpha_1}\right]$$
$$= -\left[\delta^{m_1}_{m}\phi^{m'm_2}_{\alpha_1\ \alpha_2} + \delta^{m_2}_{m}\phi^{m_1m'}_{\alpha_1\alpha_2} + (-1)^{(m+m')}\left(\delta^{m_1}_{\bar{m}'}\phi^{\bar{m}m_2}_{\alpha_1\alpha_2} + \delta^{m_2}_{\bar{m}'}\phi^{m_1\bar{m}}_{\alpha_1\alpha_2}\right)\right]. \tag{23}$$

Continuing this recursive process, the general formula (19) for the prolongation coefficients will be found. □

As the symplectic vector fields (8) do not depend on any dependent variable, then the prolongation (17) can be generalized to an arbitrary number q of dependent variables $\phi \to \phi_a, a = 1, 2, \ldots, q$, by just adding another sum to the dependent variable indices a. Naturally, equations (18) and (19) still hold for each dependent variable ϕ_a. The symplectic differential invariants are determined in the next two subsections.

The prime on the sum symbol in (17) means that the sequence of index pairs $(\alpha_1, m_1), \ldots, (\alpha_n, m_n)$ appearing in the coefficients φ must be ordered and also that repeated pairs in it must be avoided. For example, let $j = 1/2$ and $N = 1$. Then take two of all possible two-pair sequences formed by (α, m), for example, $(1, 1/2), (1, -1/2)$ and $(1, -1/2), (1, 1/2)$. Their corresponding coefficients are

$$\varphi^{m'1\bar{1}}_{m\ 11} := \varphi^{m'+\frac{1}{2}-\frac{1}{2}}_{m\quad 1\ \ 1} \qquad \varphi^{m'\bar{1}1}_{m\ 11} := \varphi^{m'-\frac{1}{2}+\frac{1}{2}}_{m\quad 1\ \ 1}. \tag{24}$$

Then only one coefficient, $\varphi^{m'1\bar{1}}_{m\ 11}$ or $\varphi^{m'\bar{1}1}_{m\ 11}$, should be taken into consideration (not both). This same remark applies to the sequence of index pairs $(m_1, \alpha_1), \ldots, (m_n, \alpha_n)$ appearing in the operators $\hat{\phi}$, (18). Note that in this example all denominators appearing in the angular components $m = \pm 1/2$ in (24) were suppressed. Also, the minus sign was turned into a bar sign, $-m = \bar{m}$. This notation is used throughout this paper.

3.1. The $N \geqslant 2j + 1$ case

In the case $N \geqslant 2j + 1$, where N is the number of different bosons, we can find one invariant frame, i.e. a set of linearly independent vector fields (see [1, definition 2.83]), and one invariant coframe, i.e. a set of linearly independent horizontal one forms (see [1, definition 5.40]), which allow us to write the invariants in a simple formula. This formula is given in the following theorem.

Theorem 2. *When $N \geqslant 2j + 1$, all strictly independent nth-order differential invariants associated with the prolonged vector fields (17) are contained in the set*

$$I^{(n)}_{\alpha_1\alpha_2\cdots\alpha_{2n}} = \prod_{i=1}^{n}\left(\sum_{m_i=j}^{-j}\eta_{\alpha_{2i-1}m_i}\right)\phi^{m_1m_2\cdots m_n}_{\alpha_2\alpha_4\cdots\alpha_{2n}}. \tag{25}$$

Proof. We can easily verify that generators (9) and (10) commute. Consequently, generators $C_{\alpha_1\alpha_2}$ given in (10) form an invariant frame of dimension N^2. Then, if $N \geqslant 2j + 1$, there is also an invariant coframe because the dimension of X (the number of independent coordinates) is $N(2j+1)$. An invariant coframe must have its dimension greater than or equal to the dimension of X. In that case, the operators

$$\hat{I}_{\alpha_1\alpha_2} = \sum_{m_1=-j}^{j} \eta_{\alpha_1 m_1} D_{\alpha_2 m_1} \tag{26}$$

where $D_{\alpha m}$ is the total derivative in the jet space (see [1, definition 4.14]), form a complete set of invariant differential operators. Since the vector fields (8) are independent of the dependent variable ϕ, then ϕ is a trivial invariant of zero order. Therefore, according to [1, theorem 5.48], the differentiated invariants

$$I^{(1)}_{\alpha_1\alpha_2} = \hat{I}_{\alpha_1\alpha_2}\phi = \sum_{m_1=-j}^{j} \eta_{\alpha_1 m_1} \phi^{m_1}_{\alpha_2} \tag{27}$$

contain a complete set of first-order differential invariants. A complete set of second-order differential invariants can be obtained from

$$\hat{I}_{\alpha_3\alpha_4} I^{(1)}_{\alpha_1\alpha_2} = \sum_{m_1,m_2=-j}^{j} \eta_{\alpha_1 m_1}\eta_{\alpha_3 m_2}\phi^{m_1 m_2}_{\alpha_2\alpha_4} + \delta_{\alpha_1\alpha_4} I^{(1)}_{\alpha_2\alpha_3} \tag{28}$$

which, up to first-order invariants, can be simplified to

$$I^{(2)}_{\alpha_1\alpha_2\alpha_3\alpha_4} = \sum_{m_1,m_2=-j}^{j} \eta_{\alpha_1 m_1}\eta_{\alpha_3 m_2}\phi^{m_1 m_2}_{\alpha_2\alpha_4}. \tag{29}$$

This recursive process leads to the general formula (25). □

The generalization of (25) to an arbitrary number q of dependent variables $\phi_a, a = 1, 2, \ldots, q$, is immediately obtained: it holds for each dependent variable ϕ_a because the differential invariant operators (26) do not depend explicitly on the dependent variables. Thus, when $q > 1$, the dependent variables are not mixed in the differential invariants.

Note that all invariants (25) are strong, i.e. $\mathrm{pr}^{(n)}\mathcal{L}^{m'}_{m} I^{(n)}_{\alpha_1\alpha_2\cdots\alpha_{2n}} = 0$ everywhere. Furthermore, they are scale invariant by a global factor on the independent variables, i.e. $I^{(n)}(\gamma x) = I^{(n)}(x)$.

As a consequence of theorem 2, $n = 0$ is the order of stabilization (see the dimensional considerations in [1, chapter 5]) whenever $N = 2j+1$. The corresponding stable orbit dimension is the algebra dimension $(2j+1)(j+1)$. Therefore, the $\mathrm{Sp}(2j+1)$ group acts locally effectively (see [1, theorem 5.11]). The maximal orbit dimensions s_n, i.e. the maximal number of independent prolonged vector fields in $J^{(n)}$ belonging to a given symplectic algebra, are shown in table 3 for the first three symplectic algebras. All orbit dimensions in table 3 were obtained by computing the rank of the matrices formed by the coefficients of the prolonged vector fields (17). Once the orbit dimension s_n is known, then the number i_n of functionally independent differential invariants of order at most n is

$$i_n = q\binom{p+n-1}{n} + p - s_n \tag{30}$$

where p is the number of independent coordinates and q is the number of dependent variables. Of course, the number j_n of strictly independent differential invariants of order n is

$$j_n = i_n - i_{n-1}. \tag{31}$$

Both quantities i_n and j_n, for $q = 1$, $j \leqslant 5/2$ and $N \leqslant 3$, are shown in table 4.

Table 3. The maximal orbit dimensions s_n of sp$(2j+1)$, $j = 1/2, 3/2, /5/2$, realized with N types of different bosons. All orbit dimensions were obtained by direct symbolic computation. In all cases, we are using only one dependent variable ($q = 1$).

	sp(2)		sp(4)			sp(6)			
N	s_0	s_1	s_0	s_1	s_2	s_0	s_1	s_2	s_3
1	2	3	4	7	10	6	11	18	21
2	3	3	7	10	10	11	18	21	21
3	3	3	9	10	10	15	21	21	21
4	3	3	10	10	10	18	21	21	21
5	3	3	10	10	10	20	21	21	21
6	3	3	10	10	10	21	21	21	21

Table 4. Number i_n of independent invariants of order at most n and number j_n of strictly independent invariants of order n. N is the number of different bosons in each realization of each algebra. In all cases, we are using only one dependent variable ($q = 1$).

	sp(2)				sp(4)				sp(6)			
N	i_0	j_1/i_1	j_2/i_2	j_3/i_3	i_0	j_1/i_1	j_2/i_2	j_3/i_3	i_0	j_1/i_1	j_2/i_2	j_3/i_3
1	1	1/2	3/5	4/9	1	1/2	7/9	20/29	1	1/2	16/18	51/69
2	2	4/6	10/16	20/36	2	5/7	36/43	120/163	2	5/7	73/82	364/446
3	4	6/10	21/31	56/87	4	11/15	78/93	364/457	4	12/16	171/187	1140/1327

Example 1 **(**$j = 1/2, N = 2$**).** Consider $j = 1/2$, i.e. the symplectic algebra sp(2), and $N = 2$, i.e. two different types of bosons. Let

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{32}$$

be the symplectic metric (1). Define the vectors

$$X_\alpha = (\eta_{\alpha 1}, \eta_{\alpha \bar{1}}) \qquad \Delta_\alpha = \nabla_\alpha \phi = \left(\phi_\alpha^1, \phi_\alpha^{\bar{1}}\right) \qquad \alpha = 1, 2 \tag{33}$$

and the Hessian matrices

$$H_{\mu\nu} = \begin{pmatrix} \phi_{\mu\nu}^{11} & \phi_{\mu\nu}^{1\bar{1}} \\ \phi_{\mu\nu}^{\bar{1}1} & \phi_{\mu\nu}^{\bar{1}\bar{1}} \end{pmatrix} \qquad H_{\mu\nu}^{\mathrm{T}} = H_{\nu\mu} \qquad \mu, \nu = 1, 2. \tag{34}$$

Thus, all invariants up to second order in (25) can be rewritten in a matrix form

$$I_1^{(0)} = \phi \qquad I_2^{(0)} = -X_1 g X_2 \qquad I_{\mu\nu}^{(1)} = X_\mu \Delta_\nu \qquad I_{\mu\alpha\beta\nu}^{(2)} = X_\mu H_{\alpha\beta} X_\nu. \tag{35}$$

The ordinary invariant $I_2^{(0)}$ in (35) was also obtained by Chacón and Moshinsky [20] (see also [38]). We can see from table 4 that the sets of invariants given in this example are maximal. Moreover, the alternative matrix form introduced here will be very useful for treating the special cases in the next section where the stabilization condition $N \geqslant 2j + 1$ has not been achieved yet. In that case, the number of differential invariants from (25) is smaller than the maximal number. Equally important is the fact that besides the invariants presented above, the alternative matrix forms

$$A_{\mu\nu} = \det H_{\mu\nu}, \quad \operatorname{tr}(g H_{\mu\nu})^k, \quad \operatorname{tr}(g H_{\mu\mu})^{2k} \tag{36}$$

are also sets of differential invariants. Of course, in the present case, these alternative invariants are not functionally independent of those given in (35). For example, the very simple invariant given by $\mathrm{tr}(gH_{12})$, which is linear in the second derivatives of ϕ, can be rewritten as

$$B_{12} = \mathrm{tr}(gH_{12}) = \phi_{12}^{1\bar{1}} - \phi_{12}^{\bar{1}1} = \frac{1}{I_2^{(0)}}\big(I_{1122}^{(2)} - I_{2121}^{(2)}\big) \tag{37}$$

where $I_{1122}^{(2)}$ and $I_{2121}^{(2)}$ are given in (35). Consequently, the relation (37) implies one relation between the Hessians H_{12} and H_{21}:

$$H_{12} - H_{21} + g\,\mathrm{tr}\,(gH_{12}) = 0. \tag{38}$$

Although the functional dependence among these invariants is not an easy task to achieve, similar relations to (38) can be found, e.g. the invariants in (36) satisfy

$$\mathrm{tr}(gH_{\mu\mu})^{2k} = (-1)^k 2A_{\mu\mu}^k \qquad B_{\mu\nu} = \mathrm{tr}(gH_{\mu\nu}) \qquad \mathrm{tr}(gH_{\mu\nu})^k = F(A_{\mu\nu}, B_{\mu\nu}) \quad \mu \neq \nu \tag{39}$$

where F is a polynomial in $A_{\mu\nu}$ and $B_{\mu\nu}$. Regardless of the fact that the alternative forms in (36) are not functionally independent of those in (35), they are important in the construction of dynamical systems. For instance, the invariants $A_{\mu\nu} = \det H_{\mu\nu}$ are null Lagrangians for some variational problem (see section 4). Other important alternative matrix forms will be presented in the next section.

As a final remark, it should be noted that the action of the Casimir operator (13) ($j = 1/2$ and $N = 2$) on the dependent variable ϕ is also a second-order invariant

$$\mathfrak{C}_{\mathrm{sp}_2(2)}\phi = 3X_1\Delta_1 + 3X_2\Delta_2 + X_1H_{11}X_1 + X_2H_{22}X_2 + 2X_2[H_{12} - g\,\mathrm{tr}(gH_{12})]X_1. \tag{40}$$

Using relation (38), this Casimir operator can be rewritten as a linear function in the previous differential invariants

$$\mathfrak{C}_{\mathrm{sp}_2(2)}\phi = 3\big(I_{11}^{(1)} + I_{22}^{(1)}\big) + I_{1111}^{(2)} + I_{2222}^{(2)} + 4I_{2121}^{(2)} - 2I_{2211}^{(2)}. \tag{41}$$

3.2. *The* $N < 2j + 1$ *case*

If $N < 2j + 1$, there is no invariant coframe in the bosonic realization of section 2. If there is no invariant coframe, then we cannot follow the previous (general) procedure to find closed formulae for the new differential invariants, although we can write them in matrix forms induced by those presented in the previous section. Having in mind that the second-order differential invariants are the most required in physics and applications where the symplectic algebra sp(6) is realized with up to three types of different bosons [17–20], which allows the explicit construction of all of their irreducible representations by polynomials in the bosonic operators, here we present matrix forms for the differential invariants of sp(2), sp(4) and sp(6) up to second order and $N \leqslant 3$. Higher-order invariants for sp(4) and sp(6) can be obtained following the lines in the next subsection. It should be mentioned that the examples presented here can be easily generalized to any symplectic algebra.

Let $H_{mu\nu}$ be the Hessian matrices with matrix elements given by

$$(H_{\mu\nu})_{mm'} = \phi_{\mu\nu}^{mm'} \qquad \mu, \nu \leqslant N \qquad m, m' \leqslant |j| \tag{42}$$

and g be the symplectic metric (1). Let X_μ be the vector formed by the independent variables $\eta_{\mu m}$ and Δ_μ the gradient vector of the dependent variable ϕ,

$$X_\mu = (\eta_{\mu j}, \ldots, \eta_{\mu\bar{j}}) \qquad \Delta_\mu = \big(\phi_\mu^j, \ldots, \phi_\mu^{\bar{j}}\big). \tag{43}$$

Table 5. Matrix form for the differential invariants up to second order ($n \leqslant 2$) and up to three types of bosons ($N \leqslant 3$) for the first three symplectic algebras sp($2j+1$), $j = 1/2, 3/2, 5/2$. The matrix forms with $n = 2$ and $N = 2, 3$ also occur as differential invariants for $N = 1$ and $n = 2$.

n	j	$N = 1$	$N = 2, 3$
0	Any	ϕ	ϕ, $X_\mu g X_\nu$ $(\mu \neq \nu)$
1	1/2	$X_1\Delta_1$	$X_\mu\Delta_\nu$
1	3/2, 5/2	$X_1\Delta_1$	$X_\mu\Delta_\nu$, $\Delta_\mu g\Delta_\nu$ $(\mu \neq \nu)$
2	Any	$X_1(H_{11}g)^k H_{11}(gH_{11})^k X_1$	$X_\mu\Delta_\nu$, $\Delta_\mu g\Delta_\nu$
		$X_1(H_{11}g)^k H_{11}(gH_{11})^k g\Delta_1$	$X_\alpha H_{\sigma\sigma'}X_{\alpha'}$, $X_\alpha H_{\sigma\sigma'}g\Delta_{\alpha'}$, $\Delta_\alpha H_{\sigma\sigma'}gX_{\alpha'}$, $\Delta_\alpha gH_{\sigma\sigma'}g\Delta_{\alpha'}$
		$\Delta_1 g(H_{11}g)^k H_{11}(gH_{11})^k g\Delta_1$	$X_\alpha(H_{\sigma\sigma'}gH_{\rho\rho'})g\Delta_{\alpha'}$, $\Delta_\alpha g(H_{\sigma\sigma'}gH_{\rho\rho'})g\Delta_{\alpha'}$
		$X_1(H_{11}g)^{2k}\Delta_1$	$\det H_{\sigma\sigma'}$, $\mathrm{tr}(gH_{\sigma\sigma})^{2k}$, $\mathrm{tr}(gH_{\sigma\sigma'})^k$

Then the independent invariants can be built from the matrix forms shown in table 5. We have also explicitly verified the relation

$$\mathfrak{C}_{\mathrm{sp}(2j+1)}\phi = \sum_{\alpha=1}^{N}[(2j+2)X_\alpha\Delta_\alpha + X_\alpha H_{\alpha\alpha}X_\alpha] + 2\sum_{\alpha'>\alpha=1}^{N} X_\alpha[H_{\alpha\alpha'} - \mathrm{tr}(gH_{\alpha\alpha'})g]X_{\alpha'} \tag{44}$$

among the Casimir operators and the differential invariants. The basic tools for writing these differential invariants, including the Casimir operators, are implemented in the algebraic computer package Killing [39, 40]. The next examples present a few remarks on these invariants.

Example 2 ($j = 1/2, N = 1$)**.** The differential invariants, all functionally independent, for sp(2), up to second order, $N = 1$, are given in table 5:

$$\begin{aligned} &I_1^{(0)} = \phi \qquad I_1^{(1)} = X_1\Delta_1 \qquad I_1^{(2)} = X_1H_{11}X_1 \\ &I_2^{(2)} = X_1H_{11}\tilde{\Delta}_1 \qquad I_3^{(2)} = \tilde{\Delta}_1H_{11}\tilde{\Delta}_1 \end{aligned} \tag{45}$$

where

$$H_{11} = \begin{pmatrix} \phi_{11}^{11} & \phi_{11}^{1\bar{1}} \\ \phi_{11}^{\bar{1}1} & \phi_{11}^{\bar{1}\bar{1}} \end{pmatrix} \qquad g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{46}$$

and

$$X_1 = (\eta_{11}, \eta_{1\bar{1}}) \qquad \Delta_1 = (\nabla_1\phi) = \left(\phi_1^1, \phi_1^{\bar{1}}\right) \qquad \tilde{\Delta}_1 = \Delta_1 g = \left(\phi_1^{\bar{1}}, -\phi_1^1\right). \tag{47}$$

There is only one more nontrivial alternative second-order matrix form

$$A_{11} = \det H_{11} \tag{48}$$

since, in this case

$$\mathrm{tr}(gH_{11})^{2k+1} = 0 \qquad \mathrm{tr}(gH_{11})^{2k} = 2(-1)^k A_{11}^k. \tag{49}$$

The remaining matrix forms in table 5 are trivial combinations (products and powers) of invariants (45) and (48), e.g.,

$$X(H_{11}g)^{2k}\Delta = (-1)^k I_1^{(1)} A_{11}^k \qquad X(H_{11}g)^{2k+1}\Delta = (-1)^{k+1} I_2^{(2)} A_{11}^k. \tag{50}$$

Of course, the alternative invariant (48) is not functionally independent of those in (45), but its functional form is not as simple as those in (49) or (50).

In fact, for sp(2) with $N = 1$, we can univocally write all differential invariants of any order in a matrix form. For example, the four third-order invariants are

$$I_1^{(3)} = X_1V_1 \qquad I_2^{(3)} = X_1V_2 \qquad I_3^{(3)} = X_1V_3 \qquad I_4^{(3)} = \tilde{\Delta}_1V_3 \tag{51}$$

where

$$V_1 = (X_1 H_1 X_1,\, X_1 H_2 X_1) \qquad V_2 = (X_1 H_1 \tilde{\Delta}_1,\, X_1 H_2 \tilde{\Delta}_1) \qquad V_3 = (\tilde{\Delta}_1 H_1 \tilde{\Delta}_1,\, \tilde{\Delta}_1 H_2 \tilde{\Delta}_1) \tag{52}$$

and H_1 and H_2 are the Hessians of ϕ_1^1 and $\phi_1^{\bar{1}}$, respectively,

$$H_1 = \begin{pmatrix} \phi_{111}^{111} & \phi_{111}^{11\bar{1}} \\ \phi_{111}^{11\bar{1}} & \phi_{111}^{1\bar{1}\bar{1}} \end{pmatrix} \qquad H_2 = \begin{pmatrix} \phi_{111}^{11\bar{1}} & \phi_{111}^{1\bar{1}\bar{1}} \\ \phi_{111}^{1\bar{1}\bar{1}} & \phi_{111}^{\bar{1}\bar{1}\bar{1}} \end{pmatrix}. \tag{53}$$

Vectors X_1 and $\tilde{\Delta}_1$ are those given in (47). The five fourth-order differential invariants can be written as

$$I_1^{(4)} = X_1 V_1 \qquad I_2^{(4)} = X_1 V_2 \qquad I_3^{(4)} = X_1 V_3 \qquad I_4^{(4)} = X_1 V_4 \qquad I_5^{(4)} = \tilde{\Delta}_1 V_4 \tag{54}$$

with

$$\begin{aligned}
V_1 &= (X_1 A_1,\, X_1 A_2) & A_i &= \left(X_1 H_i X_1,\, X_1 H_{i+1} X_1^{\mathrm{T}}\right) & i &= 1, 2 \\
V_2 &= (X_1 A_1,\, X_1 A_2) & A_i &= (X_1 H_i \tilde{\Delta}_1,\, X_1 H_{i+1} \tilde{\Delta}_1) & i &= 1, 2 \\
V_3 &= (X_1 A_1,\, X_1 A_2) & A_i &= (\tilde{\Delta}_1 H_i \tilde{\Delta}_1,\, \tilde{\Delta}_1 H_{i+1} \tilde{\Delta}_1) & i &= 1, 2 \\
V_4 &= (\tilde{\Delta}_1 A_1,\, \tilde{\Delta}_1 A_2) & A_i &= (\tilde{\Delta}_1 H_i \tilde{\Delta}_1,\, \tilde{\Delta}_1 H_{i+1} \tilde{\Delta}_1) & i &= 1, 2
\end{aligned} \tag{55}$$

and the Hessians of ϕ_{11}^{11}, $\phi_{11}^{1\bar{1}}$ and $\phi_{11}^{\bar{1}\bar{1}}$, respectively

$$H_1 = \begin{pmatrix} \phi_{1111}^{1111} & \phi_{1111}^{111\bar{1}} \\ \phi_{1111}^{111\bar{1}} & \phi_{1111}^{11\bar{1}\bar{1}} \end{pmatrix} \qquad H_2 = \begin{pmatrix} \phi_{1111}^{111\bar{1}} & \phi_{1111}^{11\bar{1}\bar{1}} \\ \phi_{1111}^{11\bar{1}\bar{1}} & \phi_{1111}^{1\bar{1}\bar{1}\bar{1}} \end{pmatrix} \qquad H_3 = \begin{pmatrix} \phi_{1111}^{11\bar{1}\bar{1}} & \phi_{1111}^{1\bar{1}\bar{1}\bar{1}} \\ \phi_{1111}^{1\bar{1}\bar{1}\bar{1}} & \phi_{1111}^{\bar{1}\bar{1}\bar{1}\bar{1}} \end{pmatrix}. \tag{56}$$

It should be noted that only the invariants $I_1^{(1)}$, $I_1^{(2)}$, $I_1^{(3)}$ and $I_1^{(4)}$ came from (25). The remaining invariants were obtained by directly solving their respective differential equations

$$\mathrm{pr}^{(n)} \mathcal{L}_m^{m'} I^{(n)} = 0 \qquad m, m' = 1, -1 \tag{57}$$

using a polynomial ansatz to speed up the resolution. The matrix forms presented here were generalized and implemented in an algebraic computer code [40, Routine `In1_S`]. Therefore, in view of theorem 2 and the results presented in this example, any complete set of differential invariants, of any order, for sp(2), realized with an arbitrary number of different types of bosons, can be analytically computed.

Example 3 ($j = 3/2, 5/2, N \leqslant 3$)**.** The differential invariants up to second order for the symplectic algebras sp(4) and sp(6), both realized with three types of different bosons at most, are given in table 5. When $N = 1$, the matrix forms with $n = 2$ and $N = 1$ in table 5 must be used in addition to other matrix forms with $n = 2$ and $N = 2, 3$ in the same table. Since all Greek indices are equal to 1 when $N = 1$, powers greater than 2, such as $X_1 (H_{11} g)^4 \Delta_1$, must be used in order to complete the sets of independent invariants. This implies huge expressions; for instance, the second-order invariant $X_1 (H_{11} g)^4 \Delta_1$ has 3222 terms. The situation gets simpler for higher number, N, of types of bosons. For example, there is no need to use those matrix forms with $n = 2$ and $N = 1$ in table 5 when $N \geqslant 2$. It should also be remarked that there are no such relations as (38) or (39) when $j \geqslant 3/2$, even when $N = 1$.

Although table 5 only presents invariants up to second order, higher-order invariants ($n \geqslant 3$) can be calculated following the procedure outlined in example 2. Naturally, all matrix forms listed in table 5 apply to any symplectic algebra.

4. Applications

We present here how the differential invariants can be used to build a toy model having Sp(2) as its dynamical group of symmetry. In fact, each second-order differential invariant in table 5 is an sp(2)-invariant Lagrangian, as it satisfies the condition $\mathrm{pr}^{(n)}\mathbf{v}_k L + L\mathrm{Div}\xi_k = 0$, for $\mathrm{Div}(\xi) = 0$, where $\xi(\eta_{\alpha m})$ are the coefficients of the vector fields (8), regardless of how many bosons are being used. In the following paragraphs, we present a few comments on these Lagrangians.

Our first goal is to find the null Lagrangians $\mathsf{E}(L) \equiv 0$, where E is the Euler operator (see [4, definition 4.3]), i.e. the Lagrangians whose Euler–Lagrange expressions $\mathsf{E}(L)$ vanish identically. A null Lagrangian is a total divergence (see [4, theorem 4.7]), which implies, by the divergence theorem, that the associated variational problem is trivial, since the integral only depends on the boundary values of ϕ. The differential invariants $\mathrm{tr}(gH_{\mu\nu})$ and $\det(H_{\mu\nu})$, where $H_{\mu\nu}$ are the Hessians given in (42) and g is the symplectic metric (1), are null Lagrangians for any symplectic algebra realized by an arbitrary number of bosons, since they are linear and quadratic homogeneous polynomials of the top-order derivatives of ϕ alone (see [41] for further details), respectively. In fact, we can verify it explicitly,

$$\begin{array}{lll} \mathsf{E}(A_{\mu\nu}) = 0 & \mathsf{E}(B_{\mu\nu}) = 0 & A_{\mu\nu} = \det H_{\mu\nu} \\ B_{\mu\nu} = \mathrm{tr}(gH_{\mu\nu}) & (\mu \neq \nu) & N \geqslant 1. \end{array} \tag{58}$$

We can easily find many new examples besides these well-known null Lagrangians. For instance, when $N = 1$, there is one null Lagrangian among the second-order invariants given in (45):

$$\mathsf{E}(X_1 H_{11} X_1) = 6 \qquad \mathsf{E}(X_1 H_{11} \tilde{\Delta}_1) = 0 \qquad \mathsf{E}(\tilde{\Delta}_1 H_{11} \tilde{\Delta}_1) = -6A_{11} \tag{59}$$

where $\tilde{\Delta}_\mu = \Delta_\mu g$ and X_μ and Δ_μ are given in (43). When $N = 2$ there are many other null Lagrangians such as the six of them among the ten canonical second-order invariants $X_\mu H_{\alpha\beta} X_\nu$ given in (35):

$$\mathsf{E}(X_\mu H_{12} X_\mu) = 0 \qquad \mathsf{E}(X_\mu H_{\alpha\alpha} X_\nu) = 0. \tag{60}$$

The Euler–Lagrange expressions for the remaining canonical second-order invariants in (35) are constants. Many other null Lagrangians, which are also linear on second derivatives of ϕ, are given by the alternative matrix forms shown in table 5:

$$\begin{array}{ll} \mathsf{E}(X_\mu H_{\alpha\alpha} \tilde{\Delta}_\mu) = 0 \quad (\mu \neq \alpha) & \mathsf{E}(X_\mu H_{\alpha\alpha} \tilde{\Delta}_\nu) = 0 \quad (\mu \neq \nu) \\ \mathsf{E}(X_\mu H_{\alpha\beta} \tilde{\Delta}_\nu) = 0 \quad (\mu \neq \nu, \alpha \neq \beta). & \end{array} \tag{61}$$

Note that the null Lagrangians (61) also have the first derivatives of ϕ besides the second derivatives present in the null Lagrangians (60). More null Lagrangians, which are quadratic on the second derivatives of ϕ, can be chosen among the alternative matrix forms given in table 5, e.g.

$$\mathsf{E}(X_1 H_{\mu\mu} g H_{\mu\mu} X_2) = 0 \qquad \mathsf{E}(\tilde{\Delta}_1 H_{\mu\mu} g H_{\mu\mu} \tilde{\Delta}_1) = 0. \tag{62}$$

Just for completeness, it is worth saying that there are only two null Lagrangians of first order

$$\mathsf{E}(X_1 \Delta_2) = 0 \qquad \mathsf{E}(X_2 \Delta_1) = 0 \qquad \mathsf{E}(\tilde{\Delta}_1 g \tilde{\Delta}_1) = 2\,\mathrm{tr} H_{12} \tag{63}$$

as the Euler–Lagrange expressions for the remaining canonical first-order invariants in (35) are constants. Because the relations

$$\tilde{\Delta}_1 H_{\mu\mu} g H_{\mu\mu} \tilde{\Delta}_1 = -(\tilde{\Delta}_1 g \tilde{\Delta}_1) A_{\mu\mu} \tag{64}$$

exist, observe that the last Euler–Lagrange expression in (63) and the last term in (62) show us explicitly that the product of non-null Lagrangians can be a null Lagrangian.

Our second goal is to present new solutions to the Helmholtz inverse problem [42]. The starting point is to find invariants satisfying the Helmholtz condition $\mathsf{D}_P^* = \mathsf{D}_P$, where D_P is the Fréchet derivative (see [4, definition 5.24]). As an example, all null invariants appearing in (58) and (60) have self-adjoint Fréchet derivatives. Note that if P is one of those null invariants ($E(P) = 0$) satisfying the Helmholtz condition, then, up to a constant multiple, $L = \phi P$ is easily seen to be a Lagrangian whose Euler–Lagrange expression is $\mathsf{E}(L) = P$. This result is in total agreement with the homotopy formula (see [4, theorem 5.92]), for our differential invariants are all homogeneous functions in the dependent variable ϕ. As another example, the following two non-null invariants also satisfy the Helmholtz condition:

$$\mathsf{E}(X_1 H_{21} \tilde{\Delta}_1) = 2\,\mathrm{tr} H_{12} \qquad \mathsf{E}(X_2 H_{12} \tilde{\Delta}_2) = -2\,\mathrm{tr} H_{12}. \tag{65}$$

Note that these two Euler–Lagrange expressions are still differential invariants.

Our final goal is to find Sp(2)-invariant evolution equations. Let L be a Lagrangian chosen among the differential invariants given in table 5 whose Euler–Lagrange expression $\mathsf{E}(L)$ is non-constant. We have verified that $\mathsf{E}(L)$ are indeed differential invariants, which means Sp(2) is a group of volume-preserving transformations on the space spanned by the independent (η) and dependent (ϕ) variables (see [41, proposition 4.5]). Indeed, as the vector fields (8) are divergenceless and their characteristics do not depend on ϕ, then Sp(2) is a volume-preserving group (see [41, infinitesimal condition 43]). Therefore, according to [33, corollary 4.6], up to a constant multiple, $\phi_t = L(x, u^{(n)})$ is an Sp(2)-invariant evolution equation, where t is an additional independent variable (denoting time).

5. Conclusions

In this study, we first reviewed a divergenceless bosonic realization for symplectic algebras sp$(2j + 1)$, with j semi-integer, over the fundamental irreducible representation and having N types of boson operators. Using the prolongation technique, (absolute) differential invariants were obtained. This technique and the present realization allowed us to find a closed expression for calculating the differential invariants (of any order) when $N \geqslant 2j + 1$. Moreover, they are all invariant by multiplying the independent variables by a constant factor (scale-invariant). Unfortunately, this same situation does not hold when $N < 2j + 1$. In the latter case, the differential invariants up to second order were explicitly presented for the symplectic algebras sp(2), sp(4) and sp(6) realized by up to three types of bosons. In particular, an algorithm was developed to compute all differential invariants in matrix forms for the symplectic algebra sp(2). This algorithm, which is very intuitive, can be further generalized to other symplectic algebras. It should be mentioned that the second-order Casimir operators are second-order polynomials in the differential invariants.

Differential invariants over the fundamental representation and their semi-products with translations were also determined in [11] to symplectic algebras realized in a different way from the present study. Here, the symplectic algebras sp$(2j + 1)$ are realized in such a way that a polynomial basis in the creation operators for any irreducible representation can be constructed by realizing them at least with a number N of different types of bosons equal to $j + 1/2$. In [11], the bosonic realization always has $N = 2j + 1$. Consequently, its ordinary and differential invariants are particular cases ($N = 2j + 1$) of those presented here.

We have also shown through explicit examples how Lagrangians as well as evolution equations, both invariant under the symplectic group, can be built from the sp(2) differential invariants. Moreover, we have added many new solutions to the Helmholtz version of the inverse problem of the calculus of variations, which has been studied by many authors (see [4, 42] and references therein): there are symplectic differential invariants satisfying the Helmholtz condition which are the Euler–Lagrange equations for some variational problems. More extended symplectic dynamical systems and their conservation laws, including sp(4) and sp(6), which can be relevant to applications in nuclear physics as well as in molecular biology, are under our consideration.

A set of symbolic procedures was implemented [40] (http://www.if.sc.usp.br/killing) in order to handle the basic tools from the prolongation theory, the bosonic realization of symplectic algebras and their differential invariants for all cases presented in this paper as well as the necessary tools for studying new cases. In particular, all examples presented here were computed and verified by symbolic computation. See also the Vessiot package (http://www.math.usu.edu/~fg_mp) for more jet space computations.

Acknowledgments

The authors would like to acknowledge Fundação de Amparo à Pesquisa do Estado de São Paulo, Brazil, for the support given to this research, as well as P J Olver for bringing [11] to our attention during the conference 'Foundations of Computational Mathematics' (FoCM-2002).

Appendix A. Bosonic realization to the sp(4) algebra

The polynomial realization (8) to sp(4) is given in table 1 for two types of bosons ($N = 2$). Also, the root system and the unique labelling to the adjoint representation are shown in table 1. The root system is shown in the first three columns in three different bases [39, 43]. The corresponding sp(4) elements $\mathcal{L}_m^{m'}$ are shown in the last two columns, where the last column is the Cartan–Weyl notation used in [44]. The fourth column shows the 'quantum numbers' of each vector of the adjoint representation.

The sp(4) algebra has three sp(2) algebras: $\mathrm{sp}_1(2) = \left\{\mathcal{L}_3^3, \mathcal{L}_3^{\bar{3}}, \mathcal{L}_{\bar{3}}^3\right\}$, $\mathrm{sp}_2(2) = \left\{\mathcal{L}_1^1, \mathcal{L}_1^{\bar{1}}, \mathcal{L}_{\bar{1}}^1\right\}$ and $\overline{\mathrm{sp}}_1(2) = \left\{\mathcal{L}_3^3, \mathcal{L}_3^1, \mathcal{L}_1^3\right\}$. The last two algebras are canonical (non-orthogonal) while the first two are orthogonal and appear in the canonical chain

$$\mathrm{sp}(4) \supset \mathrm{sp}_1(2) \oplus \mathrm{sp}_2(2). \tag{A.1}$$

The bosonic technique developed by Moshinsky *et al* [33, 34] is also powerful to construct specific irreducible representations (irreps) of any dimension. Let us consider two examples: the adjoint irrep $[2, 0]$ and the fundamental irrep $[1, 0]$. Their weight systems are given in tables 1 and 2 (first three columns), respectively. The fourth column in both tables shows the algebraic quantum numbers [44, 45] of these irreps in the canonical chain (A.1). In the fourth column, σ_i is the highest weight of $\mathrm{sp}_i(2)$ in the chain (A.1) and $h_i = \sigma_i, \sigma_i - 2, \ldots, -\sigma_i$ are its corresponding weights. The fifth column presents the corresponding polynomial basis adapted to the chain (A.1). The polynomials in the first row of each table (the polynomial of highest weight) were found by Chacón and Moshinsky [20]. Having the polynomial of highest weight, the remaining polynomials are obtained by the action of the elements $\mathcal{L}_m^{m'}$ given in table 1.

A scalar product between the basis vectors P_i and P_j can be defined as

$$P_i \cdot P_j = \langle 0| P_i^\dagger P_j |0\rangle \tag{A.2}$$

where all creation operators η must be replaced by destruction operators ξ in the Hermitian transposition

$$[P(\eta)]^\dagger = P(\xi) \tag{A.3}$$

and $|0\rangle$ is the vacuum state (or a direct product of vacuum states) defined by

$$\xi|0\rangle = 0. \tag{A.4}$$

The analytic expressions for the matrix elements of any irreducible representation adapted to the chain (A.1) are given in [44]. These analytical expressions, together with many other properties from the representation theory of simple Lie algebras, are available in the symbolic computer package Killing [39].

References

[1] Olver P J 1995 *Equivalence, Invariants, and Symmetry* (Cambridge: Cambridge University Press)

[2] Berth M and Czichowski G 2001 Using invariants to solve the equivalence problem for ordinary differential equations *Appl. Algebra Eng. Commun. Comput.* **11** 359

[3] Souriau J M 1983 Physics and geometry *Found. Phys.* **13** 133

[4] Olver P J 1993 *Applications of Lie Groups to Differential Equations (Graduate Texts in Mathematics* vol 107*)* (Berlin: Springer)

[5] Sapiro G 2001 *Geometric Partial Differential Equations and Image Analysis* (Cambridge: Cambridge University Press)

[6] Montesinos P, Gouet V, Deriche R and Pelé D 2000 Matching color uncalibrated images using differential invariants *Image Vis. Comput.* **18** 659

[7] Wybourne B G 1974 *Classical Group for Physicists* (New York: Wiley)

[8] Gilmore R 1974 *Lie Groups, Lie Algebras and Some of Their Applications* (New York: Wiley)

[9] Barut A O and Raczka R 1986 *Theory of Group Representations and Applications* (Singapore: World Scientific)

[10] Bohm A and Ne'eman Y 1988 Dynamical groups and spectrum generating algebras *Dynamical Groups and Spectrum Generating Algebras* vol 1, ed A Bohm, Y Ne'eman and A O Barut (Singapore: World Scientific) p 3

[11] Xu X P 1998 Differential invariants of classical groups *Duke Math. J.* **94** 543

[12] Rideau G and Winternitz P 1990 Nonlinear equations invariant under the Poincaré similitude, and conformal groups in 2-dimensional space-time *J. Math. Phys.* **31** 1095

[13] Fushchich W I and Yegorchenko I A 1992 2nd-order differential invariants of the rotation group $O(n)$ and of its extensions—$E(n)$, $P(1,n)$, $G(1,n)$ *Acta Appl. Math.* **28** 69

[14] Souriau J-M 1997 *Structure of Dynamical Systems. A Symplectic View of Physics (Progress in Mathematics* vol 149*)* (Basle: Birkhauser)

[15] Guillemin V and Sternberg S 1984 *Symplectic Techniques in Physics* (Cambridge: Cambridge University Press)

[16] Hofer H and Zehnder E 1994 *Symplectic Invariants and Hamiltonian Dynamics (Birkhäuser Advanced Texts)* (Basle: Birkhauser)

[17] Hornos J E M and Hornos Y M M 1993 Algebraic model for the evolution of the genetic code *Phys. Rev. Lett.* **71** 4401

[18] Hornos J E M, Hornos Y M M and Forger M 1999 Symmetry and symmetry breaking: an algebraic approach to the genetic code *Int. J. Mod. Phys.* B **13** 2795

[19] Kent R D, Schlesinger M and Wybourne B G 1998 On algebraic approaches to the genetic code *Can. J. Phys.* **76** 445

[20] Chacón E and Moshinsky M 1995 Basis of irreps of the chain of Lie algebras $sp(6) \supset sp(4) \oplus su(2) \supset su(2) \oplus su(2) \oplus su(2)$ *Topics in Theoretical Physics—A Festschrift for Paulo L Ferreira* ed V C Aguilera-Navarro, D Galetti, B M Pimentel and L Tomio (São Paulo: IFT) p 337

[21] Iachello F and Arima A 1987 *The Interacting Boson Model* (Cambridge: Cambridge University Press)

[22] Dragt A J 1991 *Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics* (College Park, MD: Center for Theoretical Physics, University of Maryland)

[23] Carvalho M J, Rowe D J, Karram S and Bahri C 2002 Optimal basis states for a microscopic calculation of intrinsic vibrational wave functions of deformed rotational nuclei *Nucl. Phys.* A **703** 167 (see references therein)

[24] Rowe D J 1985 Microscopic theory of the nuclear collective model *Rep. Prog. Phys.* **48** 1419

[25] Escher J and Draayer J P 1998 Fermion realization of the nuclear Sp(6, R) model *J. Math. Phys.* **39** 5123

[26] Rowe D J 1996 Dynamical symmetries of nuclear collective models *Prog. Part. Nucl. Phys.* **37** 265

[27] Wigner E P 1959 *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra (Pure and Applied Physics* vol 5*)* (New York: Academic)

[28] Fano U 1959 *Irreducible Tensorial Sets* (New York: Academic)

[29] Messiah A 1976 *Quantum Mechanics* (New York: Wiley)
[30] Merzbachen E 1998 *Quantum Mechanics* (New York: Wiley)
[31] Schwinger J 1965 On angular momentum *Quantum Theory of Angular Momentum* ed L C Biedenharn and H van Dam (New York: Wiley)
[32] Moshinsky M 1962 The harmonic oscillator and supermultiplet theory. (I) The single shell picture *Nucl. Phys.* **31** 384
[33] Moshinsky M 1963 Bases for the irreducible representations of the unitary groups and some applications *J. Math. Phys.* **4** 1128
[34] Moshinsky M 1968 *Group Theory and the Many-Body Problem* (New York: Gordon and Breach)
[35] Stephani H 1989 *Differential Equations. Their Solutions Using Symmetries* (Cambridge: Cambridge University Press)
[36] Bluman G W and Klumei S 1989 *Symmetries and Differential Equations* (Berlin: Springer)
[37] Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)
[38] Forger M 1998 Invariant polynomials and Molien functions *J. Math. Phys.* **39** 1107
[39] Bernardes E S 2000 Killing—an algebraic computational package for Lie algebras *Comput. Phys. Commun.* **130** 137
[40] Barbosa M S and Bernardes E S 2003 SpDifInv—differential invariants for symplectic Lie algebras: an add-on to Killing (an algebraic computational package for Lie algebras) *Comput. Phys. Commun.* submitted
[41] Olver P J, Sapiro G and Tannenbaum A 1997 Invariant geometric evolutions of surfaces and volumetric smoothing *SIAM J. Appl. Math.* **57** 176
[42] Anderson I M and Thompson G 1992 The inverse problem of the calculus of variations for ordinary differential equations *Mem. Am. Math. Soc.* **98** 473
[43] Chen Jin-Quan 1989 *Group Representation Theory for Physicists* (Singapore: World Scientific)
[44] Bernardes E S 1999 Matrix elements for the symplectic $sp(4)$ Lie algebra *J. Phys. A: Math. Gen.* **32** 6295
[45] Cerkaski M 1987 Branching rules for $Sp(2n)$ algebra reduction on the chain $Sp(2n-2) \otimes Sp(2)$ *J. Math. Phys.* **28** 989